

MILNOR'S ISOTOPY INVARIANTS AND GENERALIZED LINK HOMOTOPY

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ABSTRACT. It has long been known that a Milnor invariant with no repeated index is an invariant of link homotopy. We show that Milnor's invariants with repeated indices are invariants not only of isotopy, but also of self C_k -moves. A self C_k -move is a natural generalization of link homotopy based on certain degree k clasper surgeries, which provides a filtration of link homotopy classes.

1. INTRODUCTION

In his landmark 1954 paper [8], Milnor introduced his epinonymous higher order linking numbers, and proved that when the multi-index I had no repeated index, the $\mu(I)$ were invariants of link homotopy. In a follow up paper [9], Milnor explored some of the properties of invariants with repeated indices.

Milnor's invariants have been studied extensively since that time. For string links, these invariants are known to be finite type [1, 7], and in fact related to (the tree part of) the Kontsevich integral in a natural and beautiful way [4].

By work of Habiro [6], the finite type invariants of knots are intimately related to clasper surgery. Taking the view that link homotopy is generated by degree one clasper surgery on the link where both leaves of the clasper are on the same component, is it possible that Milnor's isotopy invariants have some relation with clasper surgery?

The answer is yes. Let us define a *self C_k -move* on a link L to be a degree k simple tree clasper surgery on L where all leaves of the clasper are on the same component. If L' is obtained from L by a sequence of self C_k -moves, we call L and L' *self C_k -equivalent*. These moves were introduced in [15], and as we have mentioned, self C_1 -equivalence is link homotopy.

For an n -component link, Milnor invariants are specified by a multi-index I , where the entries of I are chosen from $\{1, \dots, n\}$. Let $r(I)$ denote the maximum number of times that any index appears. A Milnor invariant $\mu(I)$ is called *realizable* if there exists a link L with $\mu_L(I) \neq 0$.

Our main result is the following.

Theorem 2.1 *Let $\mu(I)$ be a realizable Milnor number. Then $\mu(I)$ is an invariant of self C_k -equivalence if and only if $r(I) \leq k$.*

Notice that a self C_k -equivalence can be realized by self $C_{k'}$ -moves when $k' < k$, and thus self C_k -equivalence classes form a filtration of link homotopy classes. Moving to larger and larger k provides more and more information about the structure of isotopy classes of links.

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The classification of links up to link homotopy has been completed by Habegger and Lin [5]. However, very little is known about the structure of links under these higher order moves. Nakanishi and Ohyama have classified two component links up to self C_2 -equivalence [11, 12, 13], and it is known that boundary links are self C_2 -equivalent to the trivial link [16].

It is well known that Milnor's link homotopy invariants vanish if and only if the link is link homotopic to the unlink. However, in Example 3.3 we will produce a boundary link that is not self C_3 -equivalent to the unlink. As all Milnor invariants vanish for boundary links, this example demonstrates that self C_k -equivalence behaves very differently for $k > 1$. Much work remains to be done before we have a clear understanding of self C_k -equivalence.

2. INVARIANCE UNDER SELF C_k -MOVES

We define a self C_k -move to be surgery on a simple, tree-like degree k clasper, all of whose leaves are on the same component. Similarly, L is self C_k -equivalent to L' if L' can be obtained from L by a sequence of self C_k -moves.

Note that link homotopy is the same as self C_1 -equivalence, and the self-delta equivalence of Shibuya [14] is the same as self C_2 -equivalence.

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Theorem 2.1. *Let $\mu(I)$ be a realizable Milnor number. Then $\mu(I)$ is an invariant of self C_k -equivalence if and only if $r(I) \leq k$.*

Proof. Suppose that $r(I) \leq k$ and L is self C_k -equivalent to L' . Since Milnor's invariants are well known to be isotopy invariants, it suffices to check a single self C_k -move realized by a clasper c .

The case $k = 1$ is that of link homotopy, and was proven by Milnor in [8].

Thus, we need only consider the case $k \geq 2$. To calculate $\mu_L(I)$ (resp. $\mu_{L'}(I)$) where I has repeated indices, we may instead study the link homotopy invariants of \overline{L} ($\overline{L'}$), the link obtained by taking the appropriate number of zero framed parallels of the components of L (L') [9]. In particular, if component i of \overline{L} corresponds to component $h(i)$ of L , then $\mu_{\overline{L}}(i_1, i_2 \dots i_m) = \mu_L(h(i_1), h(i_2) \dots h(i_m))$.

Since $k \geq 2$, the self C_k -move preserves the framing of the components of L . That is, the clasper surgery carrying L to L' carries \overline{L} to $\overline{L'}$.

The self C_k -move is realized by surgery on c , that is, a clasper with $k + 1$ leaves where all of these leaves land on the same component of L . Replace L with \overline{L} .

As $r(I) \leq k$, \overline{L} has at most k copies each component of L , so each leaf of the clasper c grips at most k components of \overline{L} . Further, as each leaf of c was on the same component of L , each leaf of c holds the same components of \overline{L} .

Use the zip construction on c to produce simple claspers on \overline{L} . For a version of the zip construction without "boxes", see 4.2 of [3].

Each simple clasper has at least two leaves on the same component of \overline{L} , and so is realized by link homotopy. See Proposition 3.1.

We have shown that \overline{L} is link homotopic to $\overline{L'}$, so $\mu_{\overline{L}} = \mu_{\overline{L'}}$ for all link homotopy invariants. Hence, when $r(I) \leq k$, $\mu_L(I) = \mu_{L'}(I)$.

Suppose that $r(I) > k$. Since the Milnor invariant is realizable, by work of Cochran [2], there exists a link L where $\mu_L(I) \neq 0$, and all lower order Milnor invariants vanish. Such a link can be obtained by Bing doubling a Hopf link along a trivalent tree whose leaves are labeled by I , and then band summing components with the same label. This process is the same as a simple tree clasper surgery on the unlink, using the same tree, where leaves labeled i grasp the i th component.

Since some index i in I is repeated at least $k + 1$ times, this clasper has at least $k + 1$ leaves on component i and so can be realized by self C_k -moves. See Proposition 3.1.

As all Milnor invariants of the unlink vanish, we have that $\mu_L(I)$ is not a self C_k invariant when $r(I) > k$. □

We end this section with an example that demonstrates the power of our methods.

Self C_2 -equivalence of links is equivalent to self-delta equivalence, a relation that has been extensively studied. Much of this work relies on invariants of self C_2 -(self-delta)-equivalence that are based on the coefficients of the Conway polynomial.

Let L be the Whitehead double of the Borromean rings (equivalently the Bing double of the Whitehead link). See Figure 1. The Alexander polynomial of this link is trivial¹, but $\mu_L(123123) = 1$. Thus, L has trivial Conway polynomial, but is not self C_2 -equivalent to the unlink.

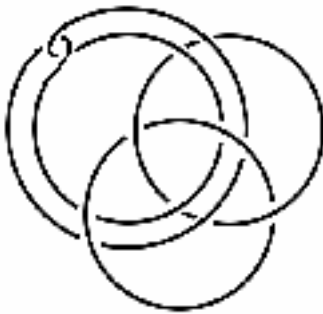


FIGURE 1. The Whitehead double of the Borromean rings.

3. FURTHER RESULTS

In the proof of Theorem 2.1 we used the fact that simple clasper surgery with multiple leaves on a single component could be converted to a self C_k -equivalence. In this section, we will find a partial converse to that fact and demonstrate its usefulness in the study of self C_k -equivalence.

We say L and L' are C_m^k -equivalent if L can be transformed into L' by ambient isotopy and degree m simple tree clasper surgery, where at least k leaves of the clasper grasp component i .

We may now state the fact mentioned above precisely.

¹J. Hillman has pointed out that the three variable Alexander polynomial of L is also 0.

Proposition 3.1. *If L is C_m^{k+1} -equivalent to L' then L is self C_k -equivalent to L' .*

Proof. We will work by induction. The base case is $m = k$, and is trivial.

Suppose now that c is a clasper representing a C_{m+1}^{k+1} move. We will show that it is C_m^{k+1} -equivalent to a C_m^{k+1} move. Begin with the clasper c . Introduce a clasper c' representing a C_m^{k+1} -move as in Figure 2 (2). We choose c' so that by reversing the zip construction, we obtain the clasper with boxes shown in Figure 2. By Move 12 of [6], we obtain the clasper c'' , which is degree m and has $k+1$ feet on component i . Thus, we can reduce any C_m^{k+1} -move to a sequence of C_k^{k+1} -moves, which is merely a self C_k -equivalence.

Figure 2 demonstrates the necessary moves. One may find it easier to read this figure from right to left. \square

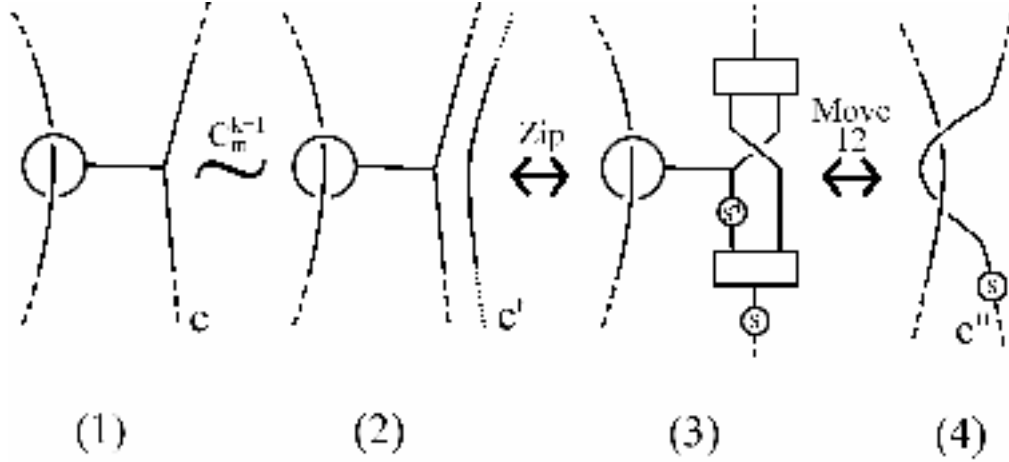


FIGURE 2. Reducing a C_{m+1}^{k+1} clasper to C_m^{k+1} claspers. The symbol s (s^{-1}) represents a positive (negative) half twist.

Proposition 3.1 is a useful, though elementary, application of the properties of claspers. Less obvious, however, is that in certain circumstances we have a converse.

Proposition 3.2. *Let L be an n component Brunnian link, and U the n component unlink. Then L is self C_k -equivalent to U if and only if L is C_{n+k-1}^{k+1} -equivalent to U .*

Proof. The ‘if’ part follows from Proposition 3.1. We will show the ‘only if’ part. The idea of this proof is similar to that of Theorem 1.2 in [10]. While in [10] they use the ‘band description’ defined in [17], we will use clasper theory.

Assume that L and U are self C_k -equivalent. Let L_i be the i th component of L . We can describe L as the unlink $O_1 \cup \dots \cup O_n$ with claspers $c_{\{i\},j}$ representing the self C_k -moves on the i th component. Since L is Brunnian, $L \setminus L_1$ is the unlink. Thus, L is obtained from the trivial link by C_k -moves corresponding to $c_{\{1\},j}$ and by crossing changes between O_1 and $c_{\{i\},j}$ ($i \geq 2$).

Passing a clasper through O_1 results in two new claspers, one of degree $k + 1$ which has a new leaf on O_1 , and one of degree k which does not. Call the former $c_{\{1i\},j}$, and the later $c'_{\{i\},j}$. See Figure 3.

We can now express L as the unlink $O_1 \cup \cdots \cup O_n$ with claspers $c_{\{1\},j}$ of degree k , claspers $c'_{\{i\},j}$ ($i \geq 2$) of degree k and claspers $c_{\{1i\},j}$ ($i \geq 2$) of degree $k + 1$.

Let L' be the link obtained by surgering the unlink $O_1 \cup \cdots \cup O_n$ only on the $c'_{\{i\},j}$. By the construction of the $c'_{\{i\},j}$, $L' = L \setminus L_1 \amalg O_1$. Since L is Brunnian, L' is the unlink, and so the claspers $c'_{\{i\},j}$ act trivially.

Thus we can express L as the unlink $O_1 \cup \cdots \cup O_n$ with claspers $c_{\{1\},j}$ of degree k and claspers $c_{\{1i\},j}$ ($i \geq 2$) of degree $k + 1$ that have leaves on $O_1 \cup O_i$.

Since $L \setminus L_2$ is the unlink, repeating the argument above shows that L is expressed as the unlink $O_1 \cup \cdots \cup O_n$ with claspers $c_{\{12\},j}$ of degree $k + 1$ that have leaves on $O_1 \cup O_2$ and claspers $c_{\{12i\},j}$ ($i \geq 3$) of degree $k + 2$ that have leaves on $O_1 \cup O_2 \cup O_i$.

Repeating these process for each L_i , finally, we can express L as the unlink $O_1 \cup \cdots \cup O_n$ with claspers $c_{\{12,\dots,n\},j}$ of degree $k + n - 1$ that have leaves on $O_1 \cup O_2 \cup \cdots \cup O_n$. Hence L is C_{k+n-1} -equivalent to the unlink. \square

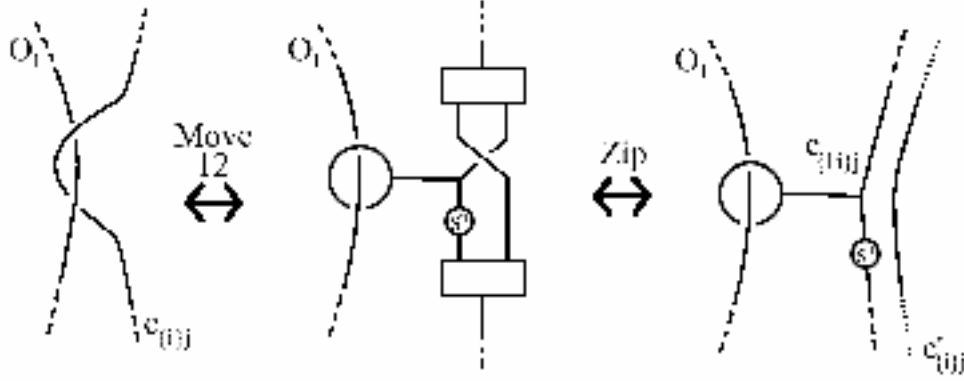


FIGURE 3. Altering the clasper $c_{\{ij\}}$ to reflect a crossing change between that clasper and O_1 .

Figure 4 illustrates the argument of Proposition 3.2 for the Whitehead link. The Whitehead link is Brunnian and link homotopic to the unlink. The center image of Figure 4 shows it as the unlink with one self C_1 clasper. The right hand image shows the C_2 -equivalence of the Whitehead link and the unlink.

Example 3.3. *The link in Figure 5 is obtained from Whitehead doubling both components of the Hopf link and is a boundary link, so all Milnor invariants vanish. However, examine the Jones polynomial.*

$$J(L) = q^{-\frac{9}{2}} - 2q^{-\frac{7}{2}} + q^{-\frac{5}{2}} - q^{-\frac{3}{2}} - q^{\frac{3}{2}} + q^{\frac{5}{2}} - 2q^{\frac{7}{2}} + q^{\frac{9}{2}}$$

We obtained the Jones polynomial from the Knot Atlas ² and by calculation using Knot ³. Evaluating the third derivative of $J(L)$ at 1, we obtain a value distinct from that of the third derivative of Jones polynomial of the unlink at 2. Since this is a

²<http://www.math.toronto.edu/~drorbn/KAtlas>, link L11n247

³<http://www.math.kobe-u.ac.jp/~kodama/knot.html>

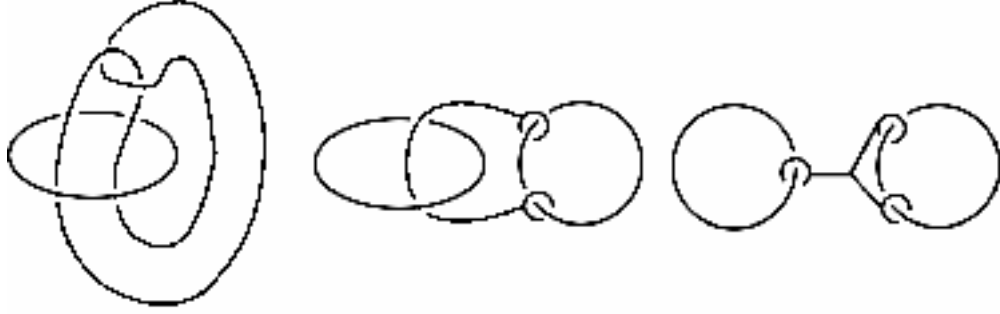


FIGURE 4. Proposition 3.2 for the Whitehead link.

finite type invariant of degree three, L is not C_4 -equivalent to the trivial link. Using Proposition 3.2 we see that it is not self C_3 -equivalent to the trivial link. In fact, since L is Brunnian, if L is self C_k -equivalent to a split link, it is self C_k -equivalent to the trivial link. Thus, L is not self C_3 -equivalent to a split link.

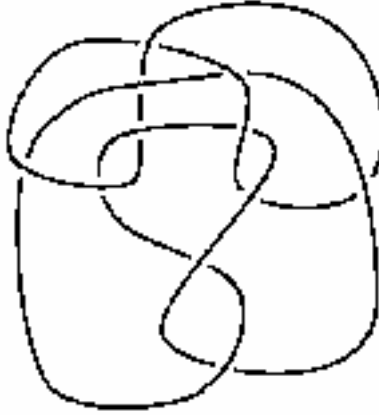


FIGURE 5. The Hopf link with both components Whitehead doubled.

Example 3.3 is interesting for other reasons. It is well known that Milnor's link homotopy invariants vanish if and only if L is link homotopic to the unlink. The example above shows that a similar statement is not true for Milnor's self C_3 invariants. While the link in example is a boundary link, it is not self C_3 -equivalent to the trivial link or even a split link. It would be interesting to know what the vanishing of Milnor's self C_k invariants implies about the self C_k -equivalence class of L .

For two component links, vanishing $\mu_L(12)$ and $\mu_L(1122)$ implies that the link is self C_2 -equivalent to the unlink [13]. Shibuya and the second author have recently shown that all boundary links are self C_2 -equivalent to the unlink [16], but whether the vanishing of Milnor's self C_2 invariants is sufficient to show that a link is self C_2 -equivalent to the unlink remains an open question.

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